

Incompressible turbulent flow on smooth surfaces

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Dedicated to the memory of our grandson Gye, that left us in January 13, 2020, when he was 28 years old.

Abstract

The article present a universal model of shear stress in turbulent flow over smooth surfaces at Reynolds number (based on the boundary layer thickness and the velocity on its edge) at range of 2×10^3 and 6.5×10^5 .

The model is based on mathematical analysis together with experimental data of turbulent flow in smooth circular pipe.

The shear stress model were used to solve the momentum equation for some surfaces and showed good correlation with the experimental data.

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List of the most commonly used symbols

- x - coordinate in the stream direction
y - coordinate perpendicular to the surface

λ - coefficient, equal to 0.0594

ρ - density

U - velocity on the edge of the boundary layer in x direction

u - velocity inside the boundary layer in x direction

v - velocity in y direction

τ - shear stress

μ - viscosity

ν - kinematic viscosity

δ - typical length in y direction

Subscript:

w - refers to the surface

∞ - refers to flow far ahead the surface

1. Derivation of the shear stress model

The fundamental to get the shear stress of turbulent flow over smooth surface, is to assume that there is a universal function that relates it to the first derivative of the main velocity and apply it on a known experimental shear stress. A mathematical analysis yields the unique possibility for a general model. Then, this model has to be used in solving the momentum equation in many cases and if the various solutions correlate with the experimental data, the model can be assumed to be a universal.

The experimental case that has been chosen is the flow in a circular pipe that was investigated very, because the shearing stress and the velocity depend only on the vertical axis y .

1.1 Flow through smooth circular pipe – experimental data

H. Schlichting (1) summarized the work, carried out by many researchers, on turbulent flow in circular pipe. The empirical results for this case are:

$$(1.1) \quad \frac{u}{U} = \left(\frac{y}{r}\right)^{\frac{1}{n}}$$

$$(1.2) \quad \frac{U_m}{U} = \frac{n^2}{(n+1) \cdot (n+0.5)}$$

$$(1.3) \quad \tau_w = \frac{\epsilon}{8} \cdot \rho \cdot U_m^2$$

$$(1.4) \quad \frac{1}{\sqrt{\epsilon}} = 2 \cdot \log\left(\frac{U_m \cdot r}{\nu} \cdot \sqrt{\epsilon}\right) - 0.2$$

Where:

r - radius of the pipe

U_m - average velocity (flow rate / pipe area)

n - exponent depends on $\frac{U_m r}{\nu}$

ϵ - coefficient of resistance

eqs. (1.1 - 1.4) yield that the best approximation to the experimental shearing stress is:

$$(1.5) \quad \frac{\tau_w}{\rho \cdot U^2} = 0.01425 \cdot \left(\frac{\nu}{U \cdot r}\right)^{\frac{1}{5}}$$

Comparison of the shearing stress in eq.(1.3) to that in eq.(1.5) is given in table 1.1

$\frac{U_m r}{\nu}$	2×10^3	1.15×10^4	5.5×10^4	5.5×10^5	1×10^6
n	6	6.6	7	8.8	10
$\frac{U r}{\nu}$	2.528×10^3	1.425×10^4	6.732×10^4	6.47×10^5	1.155×10^6
$\frac{\tau_w}{\rho U^2}$ experimental	0.00312	0.00204	0.00147	0.00103	0.000975
$\frac{\tau_w}{\rho U^2} = 0.01425 \left(\frac{\nu}{U r}\right)^{1/5}$	0.00297	0.00210	0.00154	0.00098	0.000874

Table 1.1 – shear stress in eq.(1.5) vs. experimental data

Also, the shearing stress along the vertical direction (perpendicular to the pipe axis) is:

$$(1.6) \quad \tau = \tau_w \left(1 - \frac{y}{r}\right)$$

1.2 Mathematical analysis

Under the assumption that there a general model for the shearing stress of turbulent flow over smooth surfaces we can define

$$(1.7) \quad \mu \frac{\partial u}{\partial y} = \tau \cdot \frac{\partial^2 Q}{\partial y^2}$$

Where Q is a universal function that depends only on x and y.

Since Q is a general function, eq.(1.7) valid also in the case of turbulent flow in pipe.

Thus, in this case, using eq.(1.6) yields

$$(1.8) \quad \mu \frac{\partial u}{\partial y} = \tau_w \cdot \left(1 - \frac{y}{r}\right) \cdot \frac{\partial^2 Q}{\partial y^2}$$

Integrating eq.(1.8) from y=0 to y=r yields:

$$(1.9) \quad \mu U = \tau_w \frac{Q(r,x)}{r} = 0.1425 \left(\frac{\mu}{\rho U r}\right)^{\frac{1}{5}} \rho \cdot U^2 \cdot \frac{Q(r,x)}{r}$$

This equation gives

$$(1.10) \quad Q(r,x) = \frac{1}{0.001425} \left(\frac{\mu}{\rho \cdot U}\right)^{4/5} \cdot r^{6/5}$$

The only possibility that Q would be a universal equation is if:

$$(1.11) \quad Q(y, x) = \frac{1}{0.001425} \left(\frac{\mu}{\rho \cdot U} \right)^{4/5} \cdot y^{6/5}$$

From which we get:

$$(1.12) \quad \frac{\partial^2 Q}{\partial y^2} = \frac{1}{0.0594} \left(\frac{\mu}{\rho U y} \right)^{4/5}$$

And

$$(1.13) \quad \tau = 0.0594 \left(\frac{\rho U y}{\mu} \right)^{4/5} \cdot \mu \frac{\partial u}{\partial y}$$

1.3 Principle of separated flow fields

The principle of separated flow fields states that in case of flow field over several surfaces, it is divided into sub flow fields so each surface has its own flow field. Every two close fields separated by a thin separation zone so that the velocity on both sides of it are equal and the velocity gradients become equal inside it. This principle has been used in the flow in non- circular pipe and shows a good correlation with empirical data.

2. Incompressible turbulent flow over smooth surfaces

In case of incompressible flow, eq.(1.13) can be simplified to

$$(2.1) \quad \tau = \lambda \cdot \rho \cdot \left(\frac{U y}{\nu} \right)^{4/5} \nu \frac{\partial u}{\partial y} \quad (\lambda = 0.0594)$$

3. Exact solutions of the momentum equation

3.1 Turbulent flow in smooth circular pipes

In the case of turbulent in a pipe, the shear stress in y direction is given by eq. (1.6). Comparison of this equation with eq.(2.1) yields:

$$(3.1) \quad \lambda \rho \left(\frac{Uy}{\nu} \right)^{4/5} \nu \frac{\partial u}{\partial y} = \tau_w \left(1 - \frac{y}{r} \right)$$

The solution of eq.(3.1) is:

$$(3.2) \quad \frac{u}{U} = \frac{1}{5} \left(\frac{y}{r} \right)^{1/5} \left(6 - \frac{y}{r} \right)$$

Comparison the velocity in eq.(3.2) to those in eq.(1.1) is presented in table 3.1 and shows small differences

$\left(\frac{y}{r} \right)$	0	0.2	0.4	0.6	0.8	1.0
$\frac{u}{U}$ eq.(3.2)	0	0.841	0.932	0.975	0.995	1.000
$\frac{u}{U} - \left(\frac{y}{r} \right)^{1/7}$	0	0.046	0.055	0.045	0.026	0.00
$\frac{u}{U} - \left(\frac{y}{r} \right)^{1/9}$	0	0.005	0.029	0.030	0.019	0.00

Table 3.1 – calculated velocity vs. experimental ones

And for the shear stress on the surface of the pipe we get:

$$(3.3) \quad \frac{\tau_w}{\rho U^2} = 0.01425 \left(\frac{\nu}{U \cdot r} \right)^{1/5}$$

This equation is identical to eq.(1.5)

3.2 Turbulent flow in rectangular ducts

Turbulent flow in non-circular ducts was investigated by many researchers. The experiments yield, among the rest, two importance phenomena:

- The curves of constant velocity are, principally, parallel to nearest wall. This lead to assumption of the separated flow fields that mentioned above
- The shear stress on the surface around the cross section is unified. Thus, the constant velocity curve is also a constant shear stress

Assuming a rectangular duct with walls length $2H$ and $2h$ so that $H \geq h$ as described in fig.3.1.

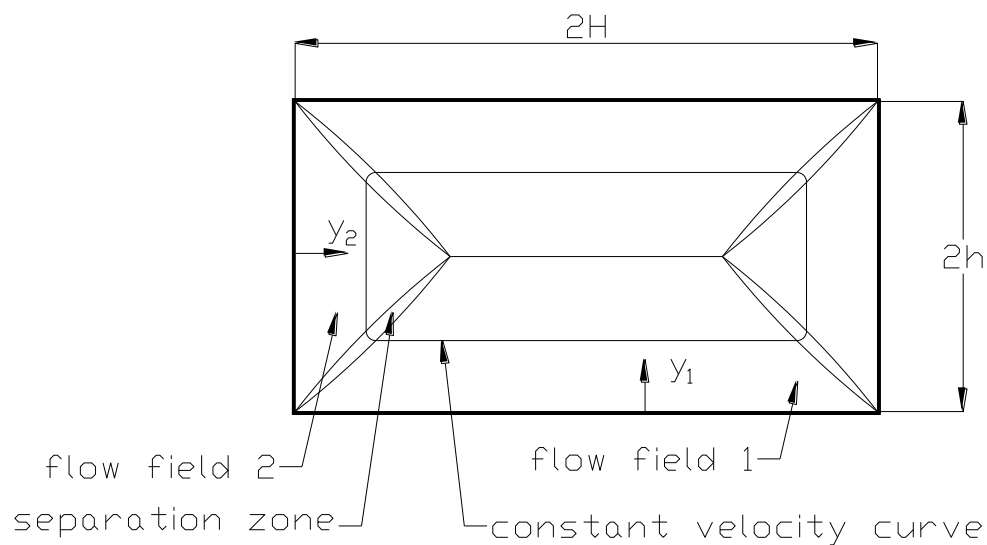


Fig. 3.1 – rectangular duct cross section

According to the principle of separated flow fields, there are two pairs of flow fields - #1 and #2- that separated by 4+1 separation zones. The 4

zones extend on the cross angles between the walls. Thus we get the following:

The distance from the walls to the constant velocity curve is the same. i.e.

$$(3.4) \quad y_1 = y_2 = y$$

The shear stress and the pressure gradient are related by

$$(3.5) \quad \tau(H + h - 2y) = \frac{\Delta P}{\Delta x} (h - y)(H - y)$$

Eq.(3.5) can be divided to 2 equations

$$(3.6) \quad \tau_w = \frac{\Delta P}{\Delta x} \frac{Hh}{(H+h)}$$

And

$$(3.7) \quad \tau = \tau_w \frac{(H-y)(h-y)}{Hh} \frac{(H+h)}{(H+h-2y)}$$

A numerical solution of eq.(3.7) together with eq.(2.1) yields, for each ratio of H/h

$$(3.8) \quad \frac{u}{U} = \frac{1}{5} \left(\frac{y}{r} \right)^{1/5} \left(6 - \frac{y}{r} \right)$$

And

$$(3.9) \quad \frac{\tau_w}{\rho U^2} = 0.01425 \left(\frac{v}{U \cdot r} \right)^{1/5}$$

3.3 Turbulent flow over horizontal flat plate with suction

Suction is one way to prevent separation and increase the lift of plane wings.

In the case of flat plate the boundary conditions are $u=0$ and $v=-V$ for $y=0$ and $u = U_\infty$ for $y = \infty$. It can be seen that the vertical velocity is independent of the length x and thus $\partial u/\partial y = 0$.

There for the momentum equation is

$$(3.10) \quad -V \frac{du}{dy} = \frac{d}{dy} \left[\lambda \left(\frac{U_\infty \cdot y}{\nu} \right)^{4/5} \cdot \nu \frac{du}{dy} \right]$$

The solution of eq.(4.21) is

$$(3.11) \quad \frac{u}{U_\infty} = 1 - \exp \left[-\frac{5}{\lambda} \cdot \frac{V}{U_\infty} \cdot \left(\frac{U_\infty \cdot y}{\nu} \right)^{1/5} \right]$$

And the shear stress is

$$(3.12) \quad \tau = U_\infty \cdot V \cdot \exp \left[-\frac{5}{\lambda} \cdot \frac{V}{U_\infty} \cdot \left(\frac{U_\infty \cdot y}{\nu} \right)^{1/5} \right]$$

These two equations enable to calculate all the required parameters.

4. Incompressible turbulent flow in boundary layer

The momentum equation in this case is:

$$(4.1) \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \lambda \frac{\partial}{\partial y} \left[\left(\frac{Uy}{\rho} \right)^{4/5} \nu \frac{\partial u}{\partial y} \right]$$

While the continuity equation yields:

$$(4.2) \quad u = \frac{\partial \phi}{\partial y} \quad v = -\frac{\partial \phi}{\partial x}$$

Assuming that the velocity profile is semi-similar in shape we can write:

$$(4.3) \quad \varphi = U\delta F(\eta, X)$$

Where

δ - Typical length in y direction

$$\eta = \left(\frac{y}{\delta}\right)^5 \quad \text{and} \quad X=x$$

Setting now

$$(4.4) \quad f = \frac{u}{U}$$

So that

$$(4.5) \quad F = \int_0^\eta 5 \cdot \eta^4 \cdot f \cdot d\eta$$

Inserting eqs.(4.2 – 4.5) into eq.(4.1) and separating variables yields 2 equations. The first one, that defines the length δ , is

$$(4.6) \quad \lambda \frac{U}{\nu} = \left(\frac{U \cdot \delta}{\nu}\right)^{1/5} \frac{d}{dx} \left(\frac{U \cdot \delta}{\nu}\right)$$

And after integration

$$(4.7) \quad \frac{U\delta}{\nu} = \left(1.2 \cdot \frac{\lambda}{\nu} \cdot \int_0^x U dX\right)^{5/6}$$

The second one, that describes the relative velocity f is

$$(4.8) \quad \begin{aligned} \frac{\partial^2 f}{\partial \eta^2} + 5F \frac{\partial f}{\partial \eta} + 25\eta^4 \frac{(U\delta)}{(U\delta)'} \cdot \frac{U'}{U} (1 - f^2) = \\ = 5 \frac{(U\delta)}{(U\delta)'} \cdot \left(5\eta^4 f \frac{\partial f}{\partial X} - \frac{\partial f}{\partial \eta} \frac{\partial F}{\partial X}\right) \end{aligned}$$

Here, the prime ' denotes differentiation in respect to X .

The boundary condition for eq.(4.8) are:

$$(4.9) \quad f(0, X) = 0 \quad \text{and} \quad f(\infty, X) = 1$$

After these transformations, the shear stress is:

$$(4.10) \quad \tau = \lambda \rho U^2 \left(\frac{\nu}{U\delta} \right)^{1/5} \frac{1}{5} \frac{\partial f}{\partial \eta}$$

4.1 Flow over flat plate parallel to the stream

In this case $U = U_\infty$

Inserting it into eq.(4.7) yields:

$$(4.11) \quad \frac{U_\infty \delta}{\nu} = \left(1.2 \frac{\lambda}{\nu} U_\infty x \right)^{5/6}$$

And since $U' = 0$ we get for a turbulent flow over the flat plate

$$(4.12) \quad \frac{d^2 f}{d\eta^2} + 5 \frac{df}{d\eta} F = 0$$

Equation (4.12) will be solved by re-integration, i.e., we assume initial f_1 by a function that fulfills as much as possible boundary conditions at

$\eta = 0$ and at $\eta = \infty$, inserting it into the equation, get f_2 and so on, until we get 2 functions, f_n and f_{n+1} that are close enough.

Since $f = \alpha\eta - \beta\eta^8 + \dots$

The first function is:

$$(4.13) \quad f_0 = a \int_0^\eta \exp(-k\eta^7)$$

a and k are calculated by the boundary condition at $\eta = \infty$

$$(4.14) \quad 1 = a \int_0^{\infty} \exp(-k\eta^7)$$

And by the momentum integral

$$(4.15) \quad 25 \int_0^{\infty} (f_0 - f_0^2) \eta^4 d\eta - a = 0$$

Eq.(4.14) and(4.15) yields

$$(4.16) \quad a = 0,984 \quad \text{and} \quad k = 0.56$$

In order to estimate f_1 was calculated by setting F_0 in eq.(4.12).

Table 4.1 shows f_0 and f_1

η	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4
f_0	0	0.197	0.394	0.589	0.776	0.924	0.991	1
f_1	0	0.197	0.395	0.591	0.777	0.925	0.991	1

Table 4.1- f_0 and f_1 vs. η

As can be seen, the deviation of f_1 compared to f_0 is less than 0.2%. thus, the relative velocity is

$$(4.17) \quad f = 0.984 \int_0^{\eta} \exp(-0.56\eta^7)$$

And

$$(4.18) \quad \frac{df(0)}{d\eta} = 0.984$$

Setting eq.(4.11) and eq.(4.18) into eq.(4.10) yields the shearing stress on the surface

$$(4.19) \quad \frac{\tau_w}{\rho U_\infty^2} = \frac{\lambda}{(1.2 \cdot \lambda)^{1/6}} \left(\frac{\nu}{U_\infty x} \right)^{1/6} \frac{0.984}{5} = 0.0182 \left(\frac{\nu}{U_\infty x} \right)^{1/6}$$

Table 4.2 shows the difference between eq.(4.19) and experimental data. (Schlichting book (1) eq.(21.12))

$\frac{U_\infty X}{\nu}$	5×10^5	1×10^6	5×10^6	1×10^7	5×10^7
$\frac{\tau_w}{\rho U_\infty^2}$ experiments	0.00215	0.00187	0.00135	0.00118	0.00085
$\frac{\tau_w}{\rho U_\infty^2}$ eq.(4.19)	0.00204	0.00182	0.00139	0.00124	(0.00095)

Table 4.2 – shear stress – eq.(4.19) vs. experimental data

4.2 Flow over two vertical flat plates (both parallel to the stream)

Assuming two vertical flat plates #1 and #2 as described in fig. 4.1

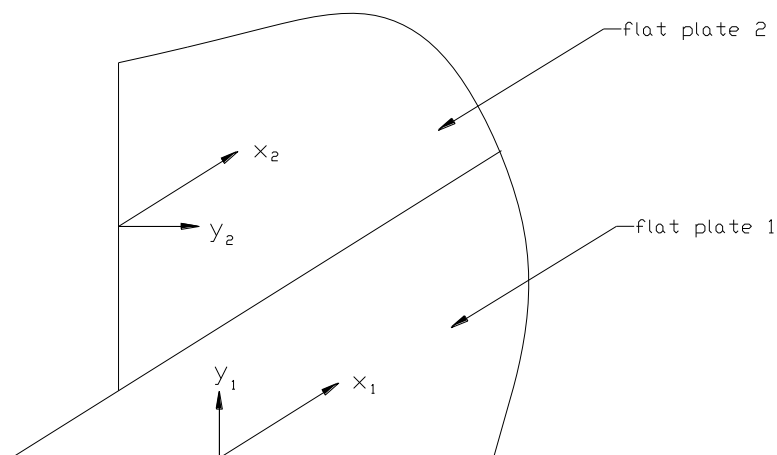


Fig. 4.1 – representation of flow over two vertical flat plates

The flow fields inside the boundary layers are

$$(4.19) \quad f_i = .984 \int_0^{\eta_i} \exp(-0.56\eta^7)$$

$$(4.20) \quad \text{Where } i=1,2$$

The separation zone extends where the velocity of the 2 fields are equal thus

$$(4.21) \quad \frac{y_1}{\delta_1} = \frac{y_2}{\delta_2}$$

And the angle between flat plate 1 and the separation θ zone, is

$$(4.22) \quad \text{tang}(\theta) = \frac{y_1}{y_2} = \frac{\delta_1}{\delta_2} = \left(\frac{x_1}{x_2}\right)^{5/6}$$

As can be seen, $\theta = 90^\circ$ in the leading edge of flat plate 2 and as x_2 increase θ decrease until it stables on the cross angle between the two flat plates.

5. Approximate solutions in 2 dimensions turbulent flow

An approximate solution can be obtained by using the integral method. This method yields a quick approximation solution without calculate step by step along the vertical axis. Instead, we assume a function for the relative velocity f that fulfills as much as possible boundary conditions on the surface and at the end of the boundary layer, and integrating the momentum equation along the vertical axis.

The boundaries conditions of the relative velocity f are:

$$(5.1) \quad \eta = 0; \quad f = 0; \quad \frac{\partial^2 f}{\partial \eta^2} = -25\eta^4 \frac{(U\delta)}{(U\delta)'} \cdot \frac{U'}{U}$$

$$(5.2) \quad \eta = \infty; \quad f = 1; \quad \frac{\partial^n f}{\partial \eta^n} = 0 \quad \text{for } n \geq 1$$

In order to fulfill the boundary conditions, together with the results of the flow over flat plate, the following profile for the relative velocity is assumed

$$(5.3) \quad f = \int_0^\eta \left(a - 5\eta^5 \frac{(U\delta)}{(U\delta)'} \cdot \frac{U'}{U} + b\eta^7 \right) \exp(-0.56\eta^7) d\eta$$

a and b are calculated by the boundary condition at $\eta=\infty$

$$(5.4) \quad b = \frac{1 - \int_0^\infty \left(a - 5\eta^5 \frac{(U\delta)}{(U\delta)'} \cdot \frac{U'}{U} \right) \exp(-0.56\eta^7) d\eta}{\int_0^\infty \eta^7 \exp(-0.56\eta^7) d\eta}$$

And by the integral of the momentum equation (4.8)

$$(5.5) \quad K + \frac{(U\delta)}{(U\delta)'} \cdot \frac{U'}{U} S + \frac{(U\delta)}{(U\delta)'} \cdot \frac{dK}{dx} - 0.04 \cdot a = 0$$

Where

$$(5.6) \quad K = \int_0^\infty (f - f^2) \eta^4 d\eta$$

And

$$(5.7) \quad S = \int_0^\infty (1 - f^2) \eta^4 d\eta$$

In order to find the initial condition it is assumed that the leading edge of the surface is a flat plate tangent to the surface as describe in fig. (5.1)

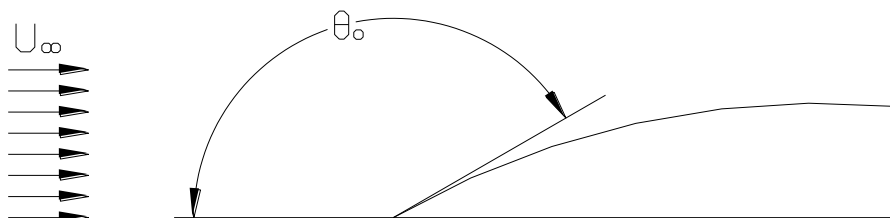


fig. 5.1 – description of the leading edge of the surface

Thus, the leading edge is calculated as a bending flat plate. i.e.

$$(5.8) \quad \frac{dK(0)}{dx} = 0$$

And the derivative of the function K at point x_{i+1} can be calculated as

$$(5.9) \quad \left(\frac{dK}{dx}\right)_{i+1} = 2 \frac{K_{i+1} - K_i}{x_{i+1} - x_i} - \left(\frac{dK}{dx}\right)_i$$

5.1 Turbulent flow over very long cylinder

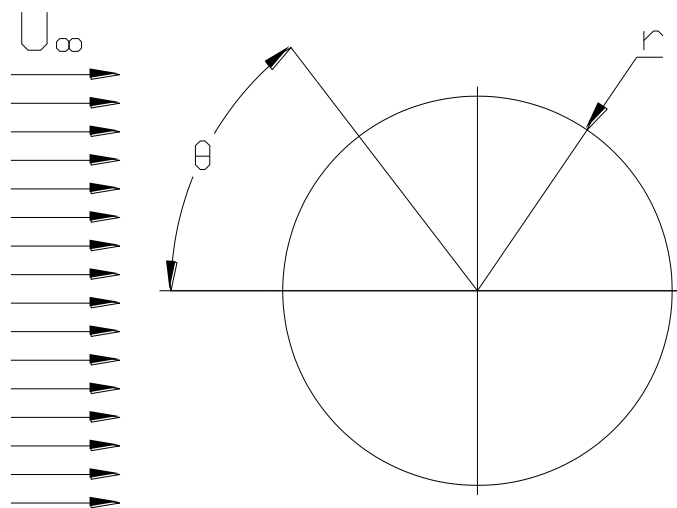


Fig. 5.2 – presentation of flow over long cylinder

In the case of flow over long cylinder the velocity U on in the edge of the boundary layer is given by

$$(5.10) \quad U = 2U_\infty \sin(\theta)$$

Where U_∞ is the velocity far ahead the cylinder and $\theta = x/r$.

Eq.(5.10) together with eq.(4.7) yield

$$(5.11) \quad \frac{(U\delta)}{(U\delta)'} \cdot \frac{U'}{U} = 1.2 \frac{\cos(\theta)}{1+\cos(\theta)}$$

The relative velocity in the boundary layer is

$$(5.11) \quad f = \int_0^\eta \left(a - 6\eta^5 \frac{\cos(\theta)}{1+\cos(\theta)} + b\eta^7 \right) \exp(-0.56\eta^7) d\eta$$

Setting now eq.(5.11) with eq.(5.4) yields the derivative of the relative velocity f' . Eq(5.11) was calculated numerically ($\theta_{i+1} - \theta_i = \pi/36$). The results are presented in table 5.1

θ°	0	15	30	45	60	75	90
a	1.270	1.266	1.254	1.231	1.185	1.137	1.041
b	1.88	1.85	1.73	1.52	1.18	0.64	-0.22

θ°	100	105	110	115	120	125	129
a	0.939	0.971	0.783	0.671	0.519	.301	0
b	-1.09	-1.65	-2.35	-3.17	-4.19	-5.40	-6.50

Table 5.1 – a and b vs. angle of location θ .

The separation angle, 129° , is similar to the separation angle that was found in experimental research that carried out by Willy Z. Sadeh Daniel Sharon and published in "NASA contractor report 3622"(2). Their results presented in fig. 5.3.

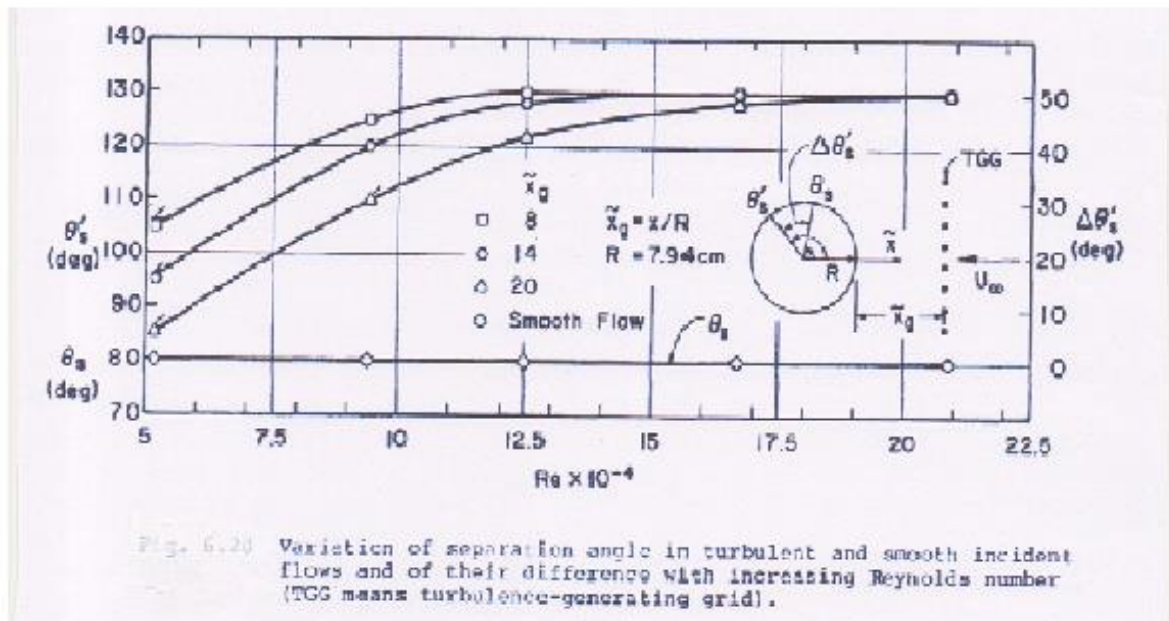


Fig. 5.3 – NASA report – separation angle of turbulent flow over cylinder

6. Conclusions

The shear stress in turbulent flow over smooth surfaces that presented here yields good results in four difference cases –flow in a circular pipe, in rectangular ducts, over a flat plate that parallel to the stream and the separation angle of flow over a circular cylinder.

Although these results are very encouraging, much more cases have to be tested until this model will be approved.

References

1. H. Schlichting – Boundary layer theory (6th edition)
2. NASA report 3622